Calculus for Nomograms

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This is a quick reference for some basic results from real analysis so I don't have to keep looking them up.

Contents

1 The implicit function theorem

Let $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be continuously differentiable, i.e., all its first partials $\partial_i f_j$ exist and are continuous. Fix a point $\langle \mathbf{a}, \mathbf{b} \rangle \in \mathbb{R}^{n+m}$ such that $F(\mathbf{a}, \mathbf{b}) = \mathbf{0}$. The Jacobian of F is:

$$JF \equiv \left[\begin{array}{ccc|ccc} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} & \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} & \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{array} \right].$$

If the $m \times m$ right-hand jacobian *submatrix* $[\partial F_i / \partial y_j]$ is invertible at $\langle \mathbf{a}, \mathbf{b} \rangle$, then there exists an open set $U \ni \mathbf{a}$ such that there is a unique "implicitly defined" function $g : U \to \mathbb{R}^m$ such that $g(\mathbf{a}) = \mathbf{b}$ and $F(\mathbf{x}, g(\mathbf{x})) = \mathbf{0}$ throughout $\mathbf{x} \in U$.

Furthermore, g is continuously differentiable, and its derivatives satisfy

$$\left[\frac{\partial g_i}{\partial x_j} \right]_{m \times n} = - \left[\frac{\partial F_i}{\partial y_j}(\mathbf{x}, g(\mathbf{x})) \right]^{-1}_{m \times m} \cdot \left[\frac{\partial F_i}{\partial x_j}(\mathbf{x}, g(\mathbf{x})) \right]_{m \times n}.$$

Corollary: The inverse function theorem

The implicit function theorem establishes the *inverse* function theorem: if you have a smooth coordinate transform $h : \mathbb{R}^m \to \mathbb{R}^m$, define the implicit function $F : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ by

$$F(y_1, \ldots, y_m; x_1, \ldots, x_m) = \langle y_1 - h_1(\mathbf{x}), y_2 - h_2(\mathbf{x}), \ldots, y_m - h_m(\mathbf{x}) \rangle.$$

The implicit function theorem informs us that, at any given point, the function mapping $y_1, \ldots, y_m$ back to $x_1, \ldots, x_m$ exists just if the Jacobian Jh has nonzero determinant there. This is the inverse function theorem, with h as an example.

In particular, the inverse of h is g, and its derivatives are defined by

$$\left[\frac{\partial g_i}{\partial y_j} \right]_{m \times m} = - \left[\frac{\partial F_i}{\partial x_j}(\mathbf{y}, g(\mathbf{y})) \right]^{-1}_{m \times m} \cdot \left[\frac{\partial F_i}{\partial y_j}(\mathbf{y}, g(\mathbf{y})) \right]_{m \times m} = + \left[\frac{\partial h_i}{\partial x_j}(\mathbf{y}) \right]^{-1}_{m \times m} \cdot \mathbf{I}_{m \times m} = (Jh)^{-1}(\mathbf{y})$$

2 The inverse function theorem

Suppose h maps an open subset $U \subset \mathbb{R}^n$ into $\mathbb{R}^n$, smoothly such that all its first partials $\partial_i h_j$ exist and are continuous (i.e., h is continuously differentiable).

If the Jacobian Jh has nonzero determinant at $a \in U$, then there exist neighborhoods $V \ni a$ and $W \ni h(a)$ such that the restriction $\widehat{h} : V \to W$ is bijective; its unique inverse function $g : W \to v$ exists and is continuously differentiable, and its derivative satisfies the matrix equation

$$Jg = (J\widehat{h})^{-1}.$$

3 The Legendre transform in one dimension

Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function of one variable. If we can find $g : \mathbb{R} \to \mathbb{R}$ which obeys the matrix equation $Df = (Dg)^{-1}$, we say that f and g are related by a Legendre transform[1].

The basic idea of a Legendre transform is that if a function's graph is nicely curved (specifically, convex), you can uniquely specify any point on it either by giving the x coordinate (which lets you find the y coordinate), or by giving the slope of the tangent line through that point (which lets you find its y-intercept, b). The Legendre transform switches between these two views, and it turns out that this process is an involution — applying it twice returns the original function.

We can use the inverse function theorem to find a formula for computing a Legendre transform of a given function.

Suppose the derivative of Df is invertible at a point a. That is, that $D^2 f \neq 0$. Then by the inverse function theorem, there exists a neighborhood of a where the inverse of Df is locally defined. Call it $\mathcal{V}(\rho)$. We will prove that a Legendre transform g of f exists, and can be defined by:

$$g(\rho) \equiv \rho \cdot \mathcal{V}(\rho) - f \circ \mathcal{V}(\rho)$$

Proof. Throughout this neighborhood, $Df \circ \mathcal{V} = \mathsf{id}$. That is, they are functional inverses of each other. (The function $\mathsf{id}(x) = x$ is the identity map, sending every number to itself.)

Instead of Df, we can compose f with $\mathcal{V}$ instead. By the chain rule, we find:

$$D(f \circ \mathcal{V}) = (Df \circ \mathcal{V}) \cdot D\mathcal{V} = \mathsf{id} \cdot D\mathcal{V}$$

The product rule gets us another way to write the right-hand side $\mathsf{id} \cdot D\mathcal{V}$, since

$$D(\mathsf{id} \cdot \mathcal{V}) = \mathcal{V} + \mathsf{id} \cdot D\mathcal{V},$$

or, by rearranging, $\mathsf{id} \cdot D\mathcal{V} = D(\mathsf{id} \cdot \mathcal{V}) - \mathcal{V}$.

Making that substitution, we find that

$$D(f \circ \mathcal{V}) = \mathsf{id} \cdot D\mathcal{V} = D(\mathsf{id} \cdot \mathcal{V}) - \mathcal{V}$$

which is another way of saying that $\mathcal{V}$ satisfies the differential equation

$$\mathcal{V}(\rho) = D(\mathsf{id} \cdot \mathcal{V} - f \circ \mathcal{V})$$

and that if there is a Legendre transform g of f such that $Dg = (Df)^{-1}$, it must satisfy the differential equation

$$Dg(\rho) = D(\mathsf{id} \cdot \mathcal{V} - f \circ \mathcal{V}).$$

[1]There's a family of such g, since adding a constant to g doesn't change the matrix equation.

In particular, by integrating both sides, we find that

$$g \equiv \mathrm{id} \cdot \mathcal{V} - f \circ \mathcal{V}$$

is one such Legendre transform whenever the local inverse $\mathcal{V}$ of Df can be defined. □

Computing the Legendre transform To *compute* the Legendre transform of $f(x)$,

1. Compute $Df(x)$, and solve the equation $\rho = Df(x)$ for $x = \mathcal{V}(\rho)$.
2. Substitute $x = \mathcal{V}(\rho)$ into the formula for f, obtaining $f \circ \mathcal{V}(\rho)$.
3. Compute a Legendre transform $g(\rho) \equiv \rho \cdot \mathcal{V}(\rho) - (f \circ \mathcal{V})(\rho)$

4 Applications to nomography

Implicit function theorem for $\widehat{w}(u,v)$. If you are making a nomogram for a relation $F(u,v,w) = 0$, and at some solution point the partial $\partial_3 F$ is nonzero, then in a neighborhood of that point, you can implicitly define $\widehat{w}(u,v)$, the 'unique value of w that solves the equation given u and v'. It has two derivatives, given by

$$\partial_i \widehat{w}(u,v) = -\frac{\partial_i F(u,v,\widehat{w}(u,v))}{\partial_3 F(u,v,\widehat{w}(u,v))} \qquad (i = 1,2)$$

The slope-intercept fields represent a smooth change of variables. If you are making a nomogram for a relation $F(u,v,w) = 0$, and you have nomographic curves $\gamma_1(u)$, $\gamma_2(v)$, $\gamma_3(w)$ which embody that relation (i.e., for each $\langle u,v,w\rangle$, we have $F(u,v,w) = 0$ if and only if the three points $\gamma_1(u)$, $\gamma_2(v)$, $\gamma_3(w)$ are collinear), you can define the slope field $A(u,v)$ and intercept field $B(u,v)$ as the slope and intercept of the isopleth line passing through $\gamma_1(u)$ and $\gamma_2(v)$.

This amounts to a map $h : \mathbb{R}^2 \to \mathbb{R}^2$, $\langle u,v\rangle \mapsto \langle A(u,v), B(u,v)\rangle$. If the Jacobian $J(A,B)$ is invertible at a point $\langle u_0, v_0\rangle$, this implies that the values of the parameters $\langle u,v\rangle$ can be uniquely recovered from the slope-intercept data $\langle A,B\rangle$ in the neighborhood of that point — that is, that h is locally invertible there.

This is a big idea because it means that either pair of numbers $\langle u,v\rangle$ or $\langle A,B\rangle$ suffices to uniquely determine the isopleth line and the (locally) unique solution defined by that line.

Miscellaneous scratchwork

Any two of the four quantities $\langle u, v, A, B\rangle$ suffices to uniquely determine the isopleth, and locally uniquely determine a solution $\widehat{w}$. An isopleth is a line drawn through the three curves $\gamma_1(u)$, $\gamma_2(v)$, $\gamma_3(w)$. (Note: although these curves may bend so that a single isopleth crosses them twice, based on the implicit function theorem we can always find a narrow enough neighborhood around a given solution to make the choice of intersection point unique.)

The two points $\gamma_1(u)$ and $\gamma_2(v)$ completely determine the isopleth line, as do its slope $A(u,v)$ and intercept $B(u,v)$. Either pair of parameters will do. And just based on the geometry of the situation, it seems clear to me that any two of the four parameters should in fact suffice to define the isopleth and therefore a local solution. (For example, fixing u defines a point in the plane, and fixing the slope A of the isopleth, now completely determines the isopleth line and the local solution.)

And indeed, suppose we are in a neighborhood of a particular solution $F(u,v,w) = 0$ where $\partial_3 F \neq 0$, so we can locally solve uniquely for $\widehat{w}(u,v)$. The two values $\langle u,v\rangle$ suffice to uniquely determine $\widehat{w}$, and also — because of the nomographic curves — define $A(u,v)$ and $B(u,v)$.

Certainly we can define a map $h(u,v,a,b) = \langle A(u,v) - a, B(u,v) - b\rangle$. The implicit function theorem says that we can solve for any two of the variables y_1, y_2 as a function of the others x_1, x_2 wherever the Jacobian submatrix of the variables we're trying to solve for is invertible.

Here's the entire Jacobian:

$$Jh \equiv \begin{bmatrix} \partial_u A(u,v) & \partial_v A(u,v) & -1 & 0 \\ \partial_u B(u,v) & \partial_v B(u,v) & 0 & -1 \end{bmatrix}$$

There are six pairs of variables we might consider, leading to six possible Jacobians.

For $\langle u,v\rangle$, the Jacobian is $J(A,B)$. For $\langle a,b\rangle$, the Jacobian is $-I$. For $\langle u,a\rangle$, it's $+\partial_u B(u,v)$. For $\langle u,b\rangle$, it's $-\partial_u A(u,v)$. For $\langle v,a\rangle$, it's $+\partial_v B(u,v)$. For $\langle v,b\rangle$, it's $-\partial_v A(u,v)$.

I believe that if any of these Jacobians vanishes, there is some degenerate nomographic funny business. This would imply that on well-behaved nomograms, these Jacobians are all always invertible and hence any two parameters out of $\langle u,v,A,B\rangle$ suffice to determine the other two.

If $\partial_u A(u,v)$ vanishes at some point, this means that $\partial_u \frac{g_1 - g_2}{f_1 - f_2}$ vanishes there, which means that $g_1'(f_1 - f_2) - (g_1 - g_2)f_1' = 0$ vanishes. Which means that the determinant $\begin{bmatrix} (f_1 - f_2) & (g_1 - g_2) \\ f_1' & g_1' \end{bmatrix}$ vanishes. This is a kind of Wrońskian determinant of $f_1 - f_2$ and $g_1 - g_2$, suggesting that they are linearly dependent (at that single point?).

It is also the following determinant

$$\partial_u A \propto \det \begin{bmatrix} f_1 & g_1 & 1 \\ f_2 & g_2 & 1 \\ f_1' & g_1' & 1 \end{bmatrix},$$

which is the Massau-determinant nomographic form of $F(u,v,w)$ with the final row $\langle f_3, g_3, 1\rangle$ replaced with $\langle f_1', g_1', 1\rangle$, for whatever that's worth.

What does the complete legendre transform of the slope-intercept fields look like? Notably, $f_1(u) = -\partial_v B/\partial_v A$ and $g_1(u) = B(u,v) - A(u,v)\cdot\partial_v B/\partial_v A$, and similarly with the roles of 1 and 2, u and v, exchanged.

So if we wanted to write $\rho = \partial_u A$ and $\sigma = \partial_v A$